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New periodic solutions for a class of singular Hamiltonian systems

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Abstract
We use the variational minimizing method to study the existence of new nontrivial periodic solutions with a prescribed energy for second order Hamiltonian systems with singular potential $V \in C^1(R^n \backslash \{0\}, R)$, which may have an unbounded potential well.

MSC: 34C15; 34C25; 58F

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1 Introduction and main results

For singular Hamiltonian systems with a fixed energy $h \in R$,

$$\ddot{q} + V'(q) = 0, \tag{1.1}$$

$$\frac{1}{2}|\dot{q}|^2 + V(q) = h. \tag{1.2}$$

Ambrosetti-Coti Zelati [1, 2] used Ljusternik-Schnirelmann theory on an C^1 manifold to get the following theorem.

Theorem 1.1 (Ambrosetti-Coti Zelati [1]) *Suppose $V \in C^2(R^n \backslash \{0\}, R)$ satisfies*

(A0)

$$V(u) \to -\infty, \quad u \to 0,$$

(A1)

$$3V'(u) \cdot u + \big(V''(u)u, u\big) \neq 0,$$

(A2)

$$V'(u) \cdot u > 0, \quad u \neq 0,$$

(A3) $\exists \alpha > 2$, *s.t.*

$$V'(u) \cdot u \leq -\alpha V(u),$$

(A4) $\exists\beta > 2, r > 0$, *s.t.*

$$V'(u) \cdot u \geq -\beta V(u), \quad 0 < |u| < r,$$

(A5)

$$\limsup_{|u|\to+\infty}\left[V(u) + \frac{1}{2}V'(u)u\right] \leq 0.$$

Then (1.1)-(1.2) *have at least one non-constant periodic solution*.

After Ambrosetti-Coti Zelati, a lot of mathematicians studied singular Hamiltonian systems. Here we only mention a related recent paper of Carminati-Sere-Tanaka [3], in which they used complex variational and geometrical and topological methods to generalize Pisani's results [5]. They got the following theorems.

Theorem 1.2 *Suppose $h > 0$, $L_0 > 0$ and $V \in C^\infty(R^n\backslash\{0\}, R)$ satisfies* (A0), (A4), *and*

(B1) $V(q) \leq 0$;

(B2) $V(q) + \frac{1}{2}V'(q)q \leq h, \forall|q| \geq e^{L_0}$;

(B3) $V(q) + \frac{1}{2}V'(q)q \geq h, \forall|q| \leq e^{-L_0}$.

Then (1.1)-(1.2) *have at least one periodic solution with the given energy h and whose action is at most $2\pi r_0$ with*

$$r_0 = \max\left\{\left[2\left(h - V(q)\right)\right]^{\frac{1}{2}}; |q| = 1\right\}.$$

Theorem 1.3 *Suppose $h > 0$, $\rho_0 > 0$ and $V \in C^\infty(R^n\backslash\{0\}, R)$ satisfies* (B1), (A4), *and*

(B2′) $\lim_{|q|\to+\infty} V'(q) = 0$;

(B3′) $V(q) + \frac{1}{2}V'(q)q \geq h, \forall|q| \leq \rho_0$.

Then (1.1)-(1.2) *have at least one periodic solution with the given energy h and whose action is at most $2\pi r_0$.*

Using variational minimizing methods, we get the following theorem.

Theorem 1.4 *Suppose $V \in C^1(R^n\backslash\{0\}, R)$ satisfies*

(V1) $\exists\alpha > 0, \beta > 2, r > 0$, *s.t.*

$$V(q) \leq -\alpha|q|^{-\beta}, \quad 0 < |q| < r;$$

(V2)

$$V(q) < 0, \quad q \neq 0;$$

(V3)

$$V(-q) = V(q), \quad q \neq 0.$$

Then for any $h > 0$, (1.1)-(1.2) *have at least one non-constant periodic solution with the given energy h*.

2 A few lemmas

Let

$$H^1 = W^{1,2}(R/Z, R^n) = \{u : R \to R^n, u \in L^2, \dot{u} \in L^2, u(t+1) = u(t)\}.$$

Then the standard H^1 norm is equivalent to

$$\|u\| = \|u\|_{H^1} = \left(\int_0^1 |\dot{u}|^2 \, dt\right)^{1/2} + \left|\int_0^1 u(t) \, dt\right|.$$

Let

$$\Lambda = \{u \in H^1 | u(t) \neq 0, \forall t\}.$$

By symmetry condition (V3), similar to Ambrosetti-Coti Zelati [1], let

$$\Lambda_0 = \{u \in H^1 = W^{1,2}(R/Z, R^n), u(t+1/2) = -u(t), u(t) \neq 0\}.$$

We define the equivalent norm in $E = \{u \in H^1 = W^{1,2}(R/Z, R^n), u(t + \frac{1}{2}) = -u(t)\}$:

$$\|u\| = \|u\|_E = \left(\int_0^1 |\dot{u}|^2 \, dt\right)^{1/2}.$$

Lemma 2.1 ([1, 4]) *Let* $f(u) = \frac{1}{2}\int_0^1 |\dot{u}|^2 \, dt \int_0^1 (h - V(u)) \, dt$ *and* $\tilde{u} \in \Lambda$ *be such that* $f'(\tilde{u}) = 0$ *and* $f(\tilde{u}) > 0$. *Set*

$$\frac{1}{T^2} = \frac{\int_0^1 (h - V(\tilde{u})) \, dt}{\frac{1}{2}\int_0^1 |\dot{\tilde{u}}|^2 \, dt}. \tag{2.1}$$

Then $\tilde{q}(t) = \tilde{u}(t/T)$ *is a non-constant* T*-periodic solution for* (1.1)-(1.2). *Furthermore, if* $V(x) < h$, $\forall x \neq 0$, *then* $f(u) \geq 0$ *on* Λ *and* $f(u) = 0$, $\forall u \in \Lambda$ *if and only if* u *is a nonzero constant.*

If $\tilde{u} \in \Lambda_0$ *such that* $f'(\tilde{u}) = 0$ *and* $f(\tilde{u}) > 0$, *then we find that* $\tilde{q}(t) = \tilde{u}(t/T)$ *is a non-constant* T*-periodic solution for* (1.1)-(1.2).

Lemma 2.2 (Gordon [6]) *Let* V *satisfy the so-called Gordon Strong Force condition*: *There exist a neighborhood* $\mathcal{N}$ *of* 0 *and a function* $U \in C^1(\Omega, \mathbb{R})$ *such that*:

(i) $\lim_{s \to 0} U(x) = -\infty$;

(ii) $-V(x) \geq |U'(x)|^2$ *for every* $x \in \mathcal{N} - \{0\}$.

Let

$$\partial\Lambda = \{u \in H^1 = W^{1,2}(R/Z, R^n), \exists t_0, u(t_0) = 0\}.$$

Then we have

$$\int_0^1 V(u) \, dt \to -\infty, \quad \forall u_n \rightharpoonup u \in \partial\Lambda.$$

Let

$$\partial\Lambda_0 = \left\{ u \in H^1 = W^{1,2}(R/Z, R^n), u\left(t + \frac{1}{2}\right) = -u(t), \exists t_0, u(t_0) = 0 \right\}.$$

Then we have

$$\int_0^1 V(u)\,dt \to -\infty, \quad \forall u_n \rightharpoonup u \in \partial\Lambda_0.$$

Lemma 2.3 *Let X be a Banach space, and let $E \subset X$ be a weakly closed subset. Suppose that $\phi(u)$ is defined on an open subset $\Lambda \subset X$ and $\phi(u) \neq -\infty$ for any $u \in \Lambda$. Let $\phi(u) = +\infty$ for $u \in \partial\Lambda$. Assume $\phi(u) \not\equiv +\infty$ and is weakly lower semi-continuous on $\bar{\Lambda} \cap E$, and that it is coercive on $\Lambda \cap E$:*

$$\phi(u) \to +\infty, \quad \|u\| \to +\infty$$

and

$$\phi(u_n) \to +\infty, \quad u_n \rightharpoonup u \in \partial\Lambda.$$

Then ϕ attains its infimum in $\Lambda \cap E$.

Proof We set

$$c = \inf_{\Lambda\cap E} \phi(u).$$

Then

$$-\infty < c < +\infty,$$

in fact, by the assumptions, it is obvious that $c < +\infty$. Now if $c = -\infty$, then there exists $\{u_n\} \subset \Lambda \cap E$ such that $\phi(u_n) \to -\infty$. Then we know that $\{u_n\}$ is bounded, since ϕ is coercive. By the Eberlein-Schmulyan theorem, $\{u_n\}$ has a weakly convergent subsequence. Finally, by the definition for c and the assumption for the weakly lower semi-continuity for $\phi(u)$, we know $\phi(u) = -\infty$. This is a contradiction.

Now we know that there exists minimizing sequence $\{u_n\}$ such that $\phi(u_n) \to c$. Furthermore by the coercivity of ϕ we know that $\{u_n\}$ is bounded; then $\{u_n\}$ has a weakly convergent subsequence. We claim the weak limit $u \in \Lambda$, since otherwise $\phi(u) = +\infty$ by the assumption. On the other hand, by the definition of the infimum c and the assumption for the weak lower semi-continuity for $\phi(u)$ on $\bar{\Lambda} \cap E$, we know $\phi(u) = c < +\infty$. This is a contradiction. So the weak limit $u \in \Lambda \cap E$ and $\phi(u) = c$. □

3 The proof of Theorem 1.4

Lemma 3.1 *Assume* (V1) *hold, then for any weakly convergent sequence $u_n \rightharpoonup u \in \partial\Lambda_0$, we have*

$$f(u_n) \to +\infty.$$

Proof Notice that (V1) imply Gordon's strong force condition. By the weak limit $u \in \partial\Lambda$ and V satisfying Gordon's strong force condition, we have

$$\int_0^1 -V(u_n)\,dt \to +\infty, \quad \forall u_n \rightharpoonup u \in \partial\Lambda.$$

By $u_n \rightharpoonup u$ in the Hilbert space H^1, we know that $\|u_n\|$ is bounded.

(1) If $u \equiv 0$, then by Sobolev's embedding theorem, we have the uniform convergence property:

$$|u_n|_\infty \to 0, \quad n \to +\infty.$$

By the symmetry of $u(t + 1/2) = -u(t)$, we have $\int_0^1 u(t)\,dt = 0$, then we have Sobolev's inequality:

$$\int_0^1 \left|\dot{u}(t)\right|^2 dt \geq 12\left|u(t)\right|_\infty^2.$$

Then we have

$$f(u_n) \geq 6|u_n|_\infty^{2-\beta} \to +\infty, \quad n \to +\infty.$$

So in this case we have

$$\liminf f(u_n) = +\infty \geq f(u).$$

(2) If $u \not\equiv 0$, then we have the following. By the weakly lower semi-continuity for the norm, we have

$$\liminf \|u_n\| \geq \|u\| > 0.$$

So, by Gordon's lemma, we have

$$\begin{aligned}\liminf f(u_n) &= \liminf\left(\frac{1}{2}\int_0^1 |\dot{u}_n|^2\,dt\right)\int_0^1 \bigl(h - V(u_n)\bigr)\,dt = +\infty \\ &\geq \frac{1}{2}\int_0^1 |\dot{u}|^2\,dt \int_0^1 \bigl(h - V(u)\bigr)\,dt = f(u).\end{aligned}$$

□

Lemma 3.2 *$f(u)$ is weakly lower semi-continuous on $\bar{\Lambda}_0$.*

Proof For any $\{u_n\} \subset \bar{\Lambda}_0 : u_n \rightharpoonup u$, by Sobolev's embedding theorem, we have uniform convergence:

$$\left|u_n(t) - u(t)\right|_\infty \to 0.$$

(i) If $u \in \Lambda_0$, then by $V \in C^1(R^n\backslash\{0\}, R)$, we have

$$\left|V\bigl(u_n(t)\bigr) - V\bigl(u(t)\bigr)\right|_\infty \to 0.$$

By the weakly lower semi-continuity for norm, we have

$$\liminf \|u_n\| \geq \|u\|.$$

Hence

$$\begin{aligned}\liminf f(u_n) &= \liminf\left(\frac{1}{2}\int_0^1 |\dot{u}_n|^2\,dt\right)\int_0^1 \bigl(h - V(u_n)\bigr)\,dt \\ &\geq \frac{1}{2}\int_0^1 |\dot{u}|^2\,dt \int_0^1 \bigl(h - V(u)\bigr)\,dt = f(u).\end{aligned}$$

(ii) If $u \in \partial\Lambda_0$, then by Λ satisfying Gordon's strong force condition, we have

$$\int_0^1 -V(u_n)\,dt \to +\infty, \quad \forall u_n \rightharpoonup u \in \partial\Lambda_0.$$

(1) If $u \equiv 0$, then

$$|u_n|_\infty \to 0, \quad n \to +\infty.$$

Then we have

$$f(u_n) \geq 6|u_n|_\infty^{2-\beta} \to +\infty, \quad n \to +\infty.$$

So in this case we have

$$\liminf f(u_n) = +\infty \geq f(u).$$

(2) If $u \not\equiv 0$. By the weakly lower semi-continuity for norm, we have

$$\liminf \|u_n\| \geq \|u\| > 0.$$

So by Gordon's lemma, we have

$$\begin{aligned}\liminf f(u_n) &= \liminf\left(\frac{1}{2}\int_0^1 |\dot{u}_n|^2\,dt\right)\int_0^1 \bigl(h - V(u_n)\bigr)\,dt = +\infty \\ &\geq \frac{1}{2}\int_0^1 |\dot{u}|^2\,dt \int_0^1 \bigl(h - V(u)\bigr)\,dt = f(u).\end{aligned}$$ □

Lemma 3.3 *$\bar{\Lambda}_0$ is a weakly closed subset of H^1.*

Proof By Sobolev's embedding theorems, the proof is obvious. □

Lemma 3.4 *The functional $f(u)$ is coercive on Λ_0.*

Proof By the definition of $f(u)$ and the assumption (V2), we have

$$f(u) = \frac{1}{2}\int_0^1 |\dot{u}|^2\,dt \int_0^1 \bigl(h - V(u)\bigr)\,dt \geq \frac{h}{2}\int_0^1 |\dot{u}|^2\,dt, \quad \forall u \in \Lambda_0.$$ □

Lemma 3.5 *The functional $f(u)$ attains the infimum on Λ_0; furthermore, the minimizer is non-constant.*

Proof By Lemma 2.2 and Lemmas 3.1-3.3, we know that the functional $f(u)$ attains the infimum in Λ_0; furthermore, we claim that

$$\inf_{\Lambda_0} f(u) > 0,$$

since otherwise, $u_0(t) = const$ attains the infimum 0, then by the symmetry of Λ_0, we have $u_0(t) \equiv 0$, which contradicts the definition of Λ_0. Now we know that the minimizer is non-constant. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
XW proved the main theorem, SS participated in the proof and helped to draft the manuscript. Both authors read and approved the final manuscript.

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